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2003 J. Phys. A: Math. Gen. 36 7989

(http://iopscience.iop.org/0305-4470/36/29/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 7989–8001 PII: S0305-4470(03)62870-2

Reversible one-dimensional cellular automata with one of the two Welch indices equal to 1 and full shifts

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Received 30 April 2003, in final form 11 June 2003
Published 8 July 2003
Online at stacks.iop.org/JPhysA/36/7989

Abstract

Reversible cellular automata are invertible discrete dynamical systems which have been widely studied both for analysing interesting theoretical questions and for obtaining relevant practical applications, for instance, simulating invertible natural systems or implementing data coding devices. An important problem in the theory of reversible automata is to know how the local behaviour which is not invertible is able to yield a reversible global one. In this sense, symbolic dynamics plays an important role for obtaining an adequate representation of a reversible cellular automaton. In this paper we prove the equivalence between a reversible automaton where the ancestors only differ at one side (technically with one of the two Welch indices equal to 1) and a full shift. We represent any reversible automaton by a de Bruijn diagram, and we characterize the way in which the diagram produces an evolution formed by undefined repetitions of two states. By means of amalgamations, we prove that there is always a way of transforming a de Bruijn diagram into the full shift. Finally, we provide an example illustrating the previous results.

PACS numbers: 02.10.Yn, 05.65.+b, 45.70.Qj, 87.18.Bb

1. Introduction

Cellular automata are discrete dynamical systems which are able to produce complex phenomena by means of simple local interactions. They were conceived by John von Neumann for constructing a self-reproducing system [19]. Other relevant works in cellular automata theory have been developed by Conway [5] and Wolfram [20].

The study of the reversible behaviour in cellular automata was first treated by Moore and Myhill [13, 15], but the main reference for the one-dimensional case is provided by the paper of Hedlund studying reversible automata as automorphisms of the full shift, establishing also the combinatorial properties fulfilled by these systems and suggesting the relation between symbolic dynamics and cellular automata theory.

Although very important results have been obtained about reversible one-dimensional cellular automata such as their detection by computational procedures [1, 7, 14], their characterization by block permutations [9] and their algebraic construction and simulation with a smaller neighbourhood size [2], few papers have analysed the relation between reversible automata and symbolic dynamics; some examples are the results by Nasu [16, 17], Kari [8] and Boyle and Maass [4].

Symbolic dynamics consists of describing a dynamical system by means of a finite set of symbols, thus the shift of sequences of symbols describes different behaviour of the system. If there is a finite set of sequences which cannot be generated by this shift then the system is a shift of finite type. A common practice in symbolic dynamics is to use graphs and matrices because they represent shift systems which are analysed in several ways. We can use the theoretical tools developed in symbolic dynamics for analysing the different evolutions produced by a one-dimensional cellular automaton, in particular, we shall study the reversible case.

The simplest kind of shift of finite type is the one where the set of forbidden sequences is empty; it is called a full shift. If the full shift is described by k states then it is a full shift of k states. A graph presentation of the full shift with k states is one node with k self-loops.

Reversible one-dimensional cellular automata are examples of systems which can generate every sequence of states, and they have a graph presentation by means of de Bruijn diagrams. Thus, the question is if we can specify some transformation from these diagrams into the corresponding full shift.

In this paper we shall investigate reversible automata where the ancestors of a given sequence of states only differ at one side, that is, reversible automata with a Welch index 1. In this sense, the goals of this paper are:

(i) To show the different kinds of paths in the de Bruijn diagram which define the sequences composed by the finite repetition of one state followed by the finite repetition of another state.
(ii) To prove that there exists a transformation from the de Bruijn diagram into the full shift of k states.

The relevance of this paper is to explain how the states are communicated to form in a single way all the sequences of states of any finite length, illustrating the transition from local behaviour to reversible global behaviour. The transformation from de Bruijn diagrams into full shifts proves that reversible one-dimensional cellular automata with a Welch index 1 are conjugated to the full shifts.

The paper is organized as follows: section 2 presents the basic concepts and the properties of reversible one-dimensional cellular automata. In particular, we expose their transformation into cellular automata of neighbourhood size 2 and their representation by de Bruijn diagrams. Section 3 provides the tools from symbolic dynamics used for studying de Bruijn diagrams. Section 4 exposes the different kinds of paths in the de Bruijn diagram which yield sequences of states of any finite length. Section 5 proves that there exists a transformation from any reversible automaton into the full shift. Section 6 gives an example of the previous results and the final section gives the concluding remarks of the paper.

2. Fundaments

A one-dimensional cellular automaton consists of a one-dimensional array of cells where each cell takes a value from a finite set K of states. The initial assignment from states to the cells of the array is the initial configuration of the automaton. The cardinality of K is represented by k and for $m \in \mathbb{Z}^+$, let K^m be the set of sequences with m cells. Let K^* be the whole set of finite sequences of cells, for $a \in K$ let a^n be the sequence in K^n formed by n repetitions of a, let a^* be the finite repetition of a and for $w \in K^*$ let w^* be the finite repetition of w. The dynamics of the automaton is given by a local mapping, for some $n \in \mathbb{Z}^+$ there is a mapping $\varphi : K^n \to K$; each sequence in K^n is a neighbourhood of the automaton, n is the neighbourhood size and the mapping φ is the evolution rule of the automaton.

The evolution rule is applied over each neighbourhood of the initial configuration at the same time, where every neighbourhood shares $n-1$ cells with the contiguous neighbourhoods at both sides. The evolution rule yields another configuration and the same process is indefinitely repeated defining a mapping between configurations of the automaton; this global mapping is the evolution of the cellular automaton. The global transition between configurations depends on the evolution rule, that is, the global behaviour is based on the local one. In this paper we shall only study finite configurations; in this case we put together the first cell of the initial configuration with the last one, and we have complete neighbourhoods for each position. The initial configuration forms a ring which yields another ring with the same number of cells by the evolution of the automaton. A one-dimensional cellular automaton with K states, neighbourhood size n and evolution rule φ is represented by $\mathcal{A} = (k, n, \varphi)$.

For a cellular automaton $A = (k, n, \varphi)$, if $w \in K^n$ and $\varphi(w) = a \in K$ then a is the evolution of w and we say that w is an ancestor of a. This ancestor has $n-1$ more cells than a, therefore for $m \in \mathbb{Z}^+$ and $w \in K^m$, the ancestors of w have $m+n-1$ cells. If some sequence $w \in K^m$ cannot be generated by φ and it may only appear in the initial configuration, then w belongs to the Garden of Eden of $\mathcal{A}$.

In order to understand the global evolution of a cellular automaton, we ought to analyse its evolution rule. In this sense, a relevant procedure which simplifies the study of $\mathcal{A} = (k, n, \varphi)$ consists of transforming $\mathcal{A}$ into another automaton $\mathcal{A}' = (k^{n-1}, 2, \tau)$. This result is presented independently by Kari [8] and Boykett [2], and we shall give only a brief description of the process.

For a cellular automaton $\mathcal{A} = (k, n, \varphi)$, take the subset $B \subseteq K^{n-1}$ such that no element in B belongs to the Garden of Eden of $\mathcal{A}$. Hence, every ancestor of each $w \in B$ has $2n-2$ cells and the whole set of ancestors of B is K^{2n-2}. Take another set S of cardinality k^{n-1}, then we can define a bijection from K^{n-1} into S. Thus, there is a bijection both from K^{2n-2} into S^2 and from B into $C \subseteq S$. By the evolution rule φ, we can define a mapping $\tau : S^2 \to C$; but τ is also the evolution rule of a cellular automaton $\mathcal{A}' = (k^{n-1}, 2, \tau)$, and we can simulate the original behaviour of $\mathcal{A}$ by another evolution rule of neighbourhood size 2. Therefore, we just need to analyse cellular automata of neighbourhood size 2 to understand the other cases.

For a cellular automaton $\mathcal{A} = (k, 2, \varphi)$, the evolution rule φ has a matrix representation M:

- The indices of M are the states in K.
- For $a, b \in K$, each entry (a, b) in M presents the neighbourhood $ab \in K^2$.
- For $a, b, c \in K$, the entry $(a, b) = c$ in M if and only if $\varphi(ab) = c$.

The matrix M induces a digraph (or directed graph) **D**:

- The nodes of **D** are the states in K.
- **D** is a complete digraph, that is, for each neighbourhood $ab \in K^2$ there is a directed edge in **D** from a to b.

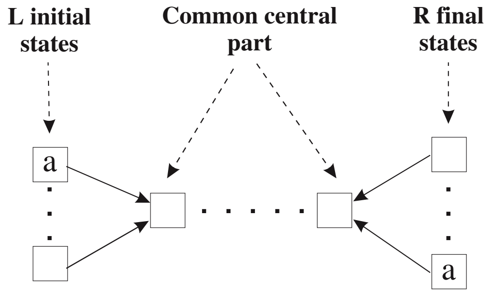

Figure 1. Ancestors in a reversible automaton; only one initial state is equal to a final one.

- Each directed edge in **D** defined by $ab \in K^2$ is labelled by $c \in K$ if and only if $(a, b) = c$ in M.

For a cellular automaton $\mathcal{A} = (k, 2, \varphi)$, **D** is the de Bruijn diagram representing the evolution rule φ [12, 18]. The labelled paths in **D** represent the sequences of states generated by φ, and the nodes forming such paths establish the ancestors of these sequences.

A special kind of cellular automaton is the one where the global mapping between configurations is invertible, that is, for $\mathcal{A} = (k, 2, \varphi)$ there exists another evolution rule φ^{-1} such that the automaton $\mathcal{A}^{-1} = (k, m, \varphi^{-1})$ (possibly $m \neq 2$) presents the inverse global behaviour of $\mathcal{A}$ [15]. Thus, $\mathcal{A}$ is reversible and φ^{-1} is the inverse evolution rule of φ.

Reversible one-dimensional cellular automata are carefully analysed by Hedlund [6]; his work presents two fundamental properties for reversible automata $\mathcal{A} = (k, 2, \varphi)$:

Property 1 (Uniform multiplicity of ancestors)**.** *For each $m \in \mathbb{Z}^+$, every sequence $w \in K^m$ has the same number of ancestors as all the other sequences, this number is equal to k.*

Property 2 (Welch indices)**.** *For some $n \in \mathbb{Z}^+$ and every $m \geqslant n$, the ancestors of each sequence $w \in K^m$ have L initial states, a common central part and R final states, fulfilling that $LR = k$. L and R are the Welch indices of the automaton.*

Property 1 says that a reversible automaton does not have Garden of Eden and property 2 specifies that the ancestors of a given sequence differ at the ends, sharing the same central sequence of states (figure 1). Welch indices show the number of different ending states at both sides for the ancestors of a given sequence. For a reversible automaton, another important property is established by Nasu in [16]:

Property 3 (Boundary condition)**.** *Let n be the minimum size of the neighbourhoods in the inverse rule, for every $m \geqslant n$ and each $w \in K^m$ there is one and only one ancestor of w with form $ava \in K^{n+1}$; $a \in K$ and $v \in K^{n-1}$.*

Property 3 defines a single ancestor for the sequence formed by the finite repetition of w, preserving the reversible behaviour of the automaton.

Let **D** be the de Bruijn diagram of a reversible automaton, then **D** has the following features by properties 1–3:

- For each $m \in \mathbb{Z}^+$ and every $w \in K^m$, there are k paths representing w in **D**.
- For some $n \in \mathbb{Z}^+$ and every $m \geqslant n$, the k paths representing $w \in K^m$ in **D** have L initial nodes, a common central node and R final nodes.
- For some $n \in \mathbb{Z}^+$ and every $m \geqslant n$, there is one and only one path representing $w \in K^m$ in **D** with the initial node equal to the final one.

D has k self-loops, each one labelled by a different state by property 3. For $a, b \in K$, a reversible automaton is able to produce the sequence a^*b^*; this sequence is relevant because it represents the path in **D** joining two self-loops, and from these kinds of paths larger sequences are generated.

For reversible automata with a Welch index 1, we shall characterize the paths a^*b^* in **D**; from this analysis we shall investigate the relation between a reversible automaton and the full shift. For this reason the next section provides the basic concepts of symbolic dynamics used for obtaining the results of this paper.

3. Symbolic dynamics

Symbolic dynamics consists of studying a dynamical system by means of discretizing both space and time. The idea is to divide the set of possible states of the system into a finite number of pieces, each represented by a symbol. Thus, we can obtain sequences of symbols representing the dynamics of the system in discrete time steps.

Using the finite set of symbols presenting the states of the system, we can now specify a digraph which shows the transitions between these states. The edges of the digraph are labelled by the states of the system, hence the paths in the digraph represent the long-term behaviour of the system; this kind of representation is an edge shift. If the edge shift has k states and it is able to produce any possible sequence of states, then the edge shift is a full shift of k states.

In this paper we shall only use simple concepts of symbolic dynamics that are taken from the book by Marcus and Lind [11]; other relevant introductions to symbolic dynamics are presented both by Kitchens [10] and Boyle [3]. An important question in symbolic dynamics is to compare different edge shifts and decide if they are equivalent. In order to resolve this question, we shall define two basic operations which transform one edge shift into another equivalent one:

Definition 1 (Elementary out-splitting)**.** *Let* **G** *be a digraph representing an edge shift, and let s, t be two nodes of* **G**. *Let E_s be the set of outgoing edges from s and let us partition E_s into n disjoint sets $E_s^1 \dots E_s^n$. An elementary out-splitting of* **G** *at node s yields a new graph* $\mathbf{G}'$ *in which s is replaced by n nodes $s_1 \dots s_n$:*

- *For $1 \leqslant i \leqslant n$, if e is an edge going from s to t in* **G** *and $e \in E_s^i$, then e goes from s_i to t in* $\mathbf{G}'$.
- *For $1 \leqslant i \leqslant n$, if e is an edge going from t to s in* **G** *then e goes from t to all the s_i nodes in* $\mathbf{G}'$.

For an elementary out-splitting the inverse operation is the following one:

Definition 2 (Elementary in-amalgamation)**.** *Let* **G** *be a digraph representing an edge shift, and let s, t be two nodes of* **G**. *Let E_s, E_t be the set of incoming edges in s and t, respectively. If the first two conditions are fulfilled, then we can perform an elementary in-amalgamation of s, t which produces a new graph* $\mathbf{G}'$ *where the nodes s, t are replaced by a single node u:*

- *Every edge in E_s has a corresponding edge in E_t with the same label and the same initial node.*
- *There is no outgoing edge from s with the same label and the same terminal node as an outgoing edge from t.*
- *If e is an incoming edge of s and t, respectively, from the same initial node in* **G**, *then e goes now from this initial node to u in* $\mathbf{G}'$.

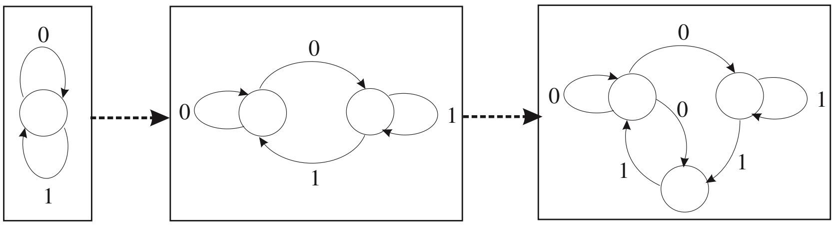

Figure 2. Out-splittings for the full shift of two states.

- *If e and e′ are distinct outgoing edges from s and t, respectively, to the same terminal node in* **G***, then both edges go now from u to this terminal node in* **G**′.

For the full shift of two states, examples of out-splittings are presented in figure 2.

Every edge shift **G** has a matrix representation M where the row and column indices of the matrix are the nodes of **G** and the entry (i, j) in M is equal to the labelled edge going from i to j in **G**. For two edge shifts $\mathbf{G}_1$ and $\mathbf{G}_2$, take their matrix representations M_1 and M_2, respectively, thus $\mathbf{G}_1$ and $\mathbf{G}_2$ have an equivalent behaviour if there exists a sequence (D_0, E_0), $(D_1, E_1), \ldots, (D_l, E_l)$ of pairs of nonnegative integral matrices such that:

$$\begin{array}{ll} M_1 = D_0E_0 & E_0D_0 = A_1 \\ A_1 = D_1E_1 & E_1D_1 = A_2 \\ A_2 = D_2E_2 & E_2D_2 = A_3 \\ & \vdots \\ A_l = D_lE_l & E_lD_l = M_2 \end{array} \tag{1}$$

in this case we say that $\mathbf{G}_1$ and $\mathbf{G}_2$ are strong shift equivalent with lag l. If two edge shifts $\mathbf{G}_1$ and $\mathbf{G}_2$ are strong shift equivalent then we can transform $\mathbf{G}_1$ into $\mathbf{G}_2$ by means of out-splittings and in-amalgamations [11]. In this paper we shall take a de Bruijn diagram associated with a reversible automaton as an edge shift which may generate each possible sequence of states.

For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ the de Bruijn diagram has k self-loops, one for each state of K. For $a, b \in K$, the paths between the self-loops of **D** must establish a single way for producing the sequence a^*b^*. In the next sections two fundamental questions shall be treated:

(i) How are the self-loops of **D** communicated?
(ii) How can we establish a strong shift equivalence between **D** and the full shift of k states?

4. Communication between states

For a reversible automaton, each node of the de Bruijn diagram has a self-loop labelled by some state in K. The self-loop labelled by $a \in K$ is the unique part of **D** able to generate the sequence a^*. For simplicity, the self-loop labelled by a will be referred to as the self-loop a.

Suppose that some node defines the self-loop a and another node has the self-loop b, hence the sequence a^*b^* must be produced in **D** by a single path going from the self-loop a to the self-loop b. More paths with these characteristics imply ancestors with several common central states for a^*b^*, therefore the automaton could not be reversible. For a^*b^* we classify

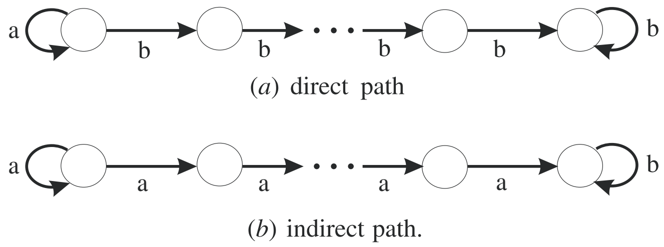

Figure 3. Paths forming the sequence a^*b^*.

a path between the self-loops a, b in **D** as follows:

- *Direct path.* If there is a sequence of edges from a to b all labelled by b.
- *Indirect path.* If there is a sequence of edges from a to b all labelled by a.

In other words, a direct path joins the self-loop a with the self-loop b by means of b^* and an indirect path by means of a^* (figure 3).

In order to characterize the paths a^*b^* defined in the de Bruijn diagram of a reversible automaton with a Welch index 1, we shall take a relevant result by Kari who establishes equivalence relations between the states of the automaton [8].

For a reversible automaton with Welch index $L = 1$, there is no pair of edges with the same label in **D** beginning from two different nodes and finishing into the same node. Therefore, each column in M is a permutation of K and for $a, b, c \in K$, if $\varphi(ab) = c$ then we define a function ϕ as $\phi_c(b) = a$, where $\phi_c(b)$ presents the initial node connected with b by an edge labelled by c in **D**. The function ϕ may be defined for sequences with more states, for $n \geqslant 2$ and $w \in K^n$; $\phi_w(b) = a$ means that there is a path labelled by w in **D** from node a to node b. Using ϕ we can establish an equivalence relation ρ_n between the states of K; two states a, b belong to ρ_n if and only if $\phi_w(a) = \phi_w(b)$ for all $w \in K^n$. For ρ_0 we take λ as the empty sequence and we define that $\phi_\lambda(a) = a$ for each $a \in K$.

A reversible automaton with Welch index $L = 1$ holds that there exists $n \in \mathbb{Z}^+$ such that the paths of length $m \geqslant n$ labelled by $w \in K^m$ begin from a single node and finish into all the nodes in **D** by property 2. Therefore, ρ_n has only one equivalence class formed by all the states in K; based on this remark, Kari proves the following result [8]:

Lemma 1. *For $i \in \mathbb{Z}^+$ and a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with Welch indices $L = 1$ and $R = k$, the equivalence relation ρ_i holds that:*

- *ρ_0 has k equivalence classes.*
- *$\rho_i \subseteq \rho_{i+1}$.*
- *$\rho_i \neq \rho_{i+1}$ if ρ_i has more than one equivalence class.*

From lemma 1 we note that the number of equivalence classes in ρ_i decreases as i grows, and another important result is defined:

Theorem 1. *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with Welch indices $L = 1$ and $R = k$, the de Bruijn diagram* **D** *has only indirect paths joining the self-loops.*

Proof. We know that all the states in $\mathcal{A}$ belong to the same equivalence class in ρ_{k-1} by lemma 1. Thus, all the paths labelled by a sequence $w \in K^{k-1}$ in **D** go from one single node to k distinct nodes.

Therefore, for $a, b \in K$, the path labelled by a^{k-1} goes from the node with the self-loop a to all the other nodes in **D**. The initial node must be the one defining the self-loop a, otherwise there are two different equivalence classes in ρ_{k-1}, which is not possible. Thus the path a^*b^* going from the self-loop a to the self-loop b is an indirect path. □

With these results we can define a computational procedure for reviewing the existence of indirect paths analysing the matrix M, take all the entries (i, j) in M such that $i \neq j$.

(i) If $(i, j) = (i, i)$ then there is an indirect path from the self-loop $a = (i, i)$ to the self-loop $b = (j, j)$.
(ii) If an indirect path from the self-loop $a = (i, i)$ to the self-loop $b = (j, j)$ has been defined, then look for an entry $(j, m) = a$ to establish an indirect path from the self-loop a to the self-loop $c = (m, m)$. Repeat the step recursively.

For a cellular automaton, this procedure is useful to establish a set of restrictions which show irreversible behaviour:

(i) There is more than one indirect path from the self-loop a to the self-loop b.
(ii) Not all the indirect paths are defined for all the ordered pairs of self-loops.

For a reversible automaton with Welch index $R = 1$, an analogous result is established taking $\phi_c(a) = b$ if and only if $\varphi(ab) = c$. Using now the previous analysis for reversible automata with Welch index $R = 1$, we obtain the following result:

Corollary 1. *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with Welch indices $L = k$ and $R = 1$, the de Bruijn diagram* **D** *has only direct paths joining the self-loops.*

Indirect paths between self-loops are relevant to know if a cellular automaton is reversible, but a better characterization shall be presented in the next section where we analyse the relation between reversible automata and full shifts.

5. Amalgamations and reversibility

In addition to the results about the paths between self-loops, we can also use lemma 1 to define a strong shift equivalence between reversible automata and full shifts.

Theorem 2. *A cellular automaton $\mathcal{A} = (k, 2, \varphi)$ is reversible with Welch index $L = 1$ if and only if every column of M is a permutation of K and there is a sequence of n in-amalgamations transforming* **D** *into the full shift of k elements.*

Proof. Let $\mathcal{A}$ be reversible with Welch index $L = 1$; by lemma 1 there are at least two states $a, b \in K$ which belong to the same equivalence class in ρ_1, hence the incoming edges going from each node to a, b have the same label in **D**.

The nodes a, b have different outgoing edges to another node because $\mathcal{A}$ has Welch index $L = 1$. Thus, a, b have the same incoming edges from each node and different outgoing edges to each node, and we can apply an in-amalgamation over a, b.

For ρ_2 we have that $a, b \in K$ are in the same equivalence class whether they also belong to the same class in ρ_1 or for the same incoming edge, a, b are connected with two distinct nodes $a', b' \in K$ which belong to the same equivalence class in ρ_1.

But in the latter case, a', b' form a new node u by the in-amalgamation defined by ρ_1; Thus, both nodes a, b have the same incoming edge from the amalgamated node u. Therefore, a, b have the same incoming edges from each node and different outgoing edges to each node because $L = 1$, and ρ_2 defines the in-amalgamation of a, b.

By induction, if ρ_n defines an in-amalgamation in **D**, for ρ_{n+1} the nodes a, b belong to the same equivalence class whether a, b also belong to the same class in ρ_n or there are two different nodes a', b' which belong to the same class in ρ_n and are connected with a, b, respectively, with the same edge, hence we can amalgamate a, b.

Finally, ρ_{k-1} has a single equivalence class where all the nodes in **D** are amalgamated into a single node with k self-loops, representing a full shift of k symbols.

On the other hand, if each column of M is a permutation of K and there is a sequence of n in-amalgamations transforming **D** into the full shift of k symbols, then in M there are at least two equal columns a, b such that $\phi_w(a) = \phi_w(b)$ for each $w \in K$ and a, b belong to the same equivalence class in ρ_1. Following the in-amalgamations, for $1 \leqslant i \leqslant n$ we have that the amalgamated matrix M_i has at least two identical columns a, b. If row u in M_i represents two or more amalgamated states, then the elements of u are identical columns in M_{i-1}, and $\phi_w(a') = \phi_w(b')$ for a', b' amalgamated states in u and $w \in K$, hence $\phi_w(a) = \phi_w(b)$ for all $w \in K^2$ where a, b are identical columns in M_i.

By induction, the same property is fulfilled for larger lengths, where $\phi_w(a) = \phi_w(b)$ for all $w \in K^m$ with $m \leqslant n$. Since M_n is formed by a single row and a single column presenting the equivalence class composed by all the states in K, each pair a, b of states fulfils that $\phi_w(a) = \phi_w(b)$ for all $w \in K^n$.

Take K^{2n}, every element in K^{2n} is represented as wv where $w, v \in K^n$, hence for each pair a, b of states we have that $\phi_v(a) = \phi_v(b)$. If $\phi_v(a) = c \in K$ then c is the common central state in the ancestors of $wv \in K^{2n}$, therefore the original evolution rule φ has an inverse evolution rule $\varphi^{-1} : K^{2n} \to K$ and $\mathcal{A}$ is reversible.

All the ancestors of $wv \in K^{2n}$ have the same left state as we can see by the in-amalgamations, hence $\mathcal{A}$ has Welch indices $R = k$ and $L = 1$. □

From theorem 2 we obtain an analogous property for reversible automata with Welch index $R = 1$.

Corollary 2. *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with Welch index $R = 1$, there exists a sequence of out-amalgamations transforming* **D** *into the full shift of k states.*

Theorem 2 shows that M is an adequate tool for detecting the in-amalgamations transforming **D** into the full shift. For a reversible automaton with Welch index $L = 1$, two states $a, b \in K$ belong to the same equivalence class in ρ_1 if a, b are identical columns and distinct rows in every position of M. In this case we amalgamate a, b defining a new matrix M_1 where the states a, b appear as a single index u, row u is the union of the entries in rows a, b in M and column u just keeps the entries of column a in M.

Thus, M is repeatedly transformed into a set of matrices of smaller order to reach a matrix of order one. Using this feature we shall prove that there are always two matrices D, E such that $DE = M$ and $ED = M_1$ where M_1 presents the in-amalgamation of **D**.

Theorem 3. *For a reversible automaton $\mathcal{A} = (k, 2, \varphi)$ with Welch index $L = 1$, there are two matrices D, E such that $DE = M$ and $ED = M_1$.*

Proof. Take D as the matrix with the row indices of M and the column indices of M_1. Each column a in D is equal to the column in M corresponding with any of the states belonging to the class represented by index a in M_1.

Take E as the matrix with the row indices of M_1 and the column indices of M, the entry (i, j) in E is defined as follows:

$$(i, j) = \begin{cases} 1 & \text{if } j \text{ belongs to the class represented by } i \text{ in } M_1. \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Table 1. Matrix M for the automaton $\mathcal{A} = (4, 2, \varphi)$.

	0	1	2	3
0	0	0	0	1
1	1	1	1	0
2	3	3	2	2
3	2	2	3	3

Thus, the product DE has the same row and column indices that M and each column i in E just takes the elements in D forming the column associated with its class, that is, i forms the original column i in M, therefore $DE = M$.

On the other hand, the product ED has the same row and column indices as M_1; row i in E shows what states belong to the equivalence class represented by i. Thus, row i in E just takes the elements of the class i for each column in D. In this case, the product of each row and every column is a copy of all the distinct elements, without making a linear combination of them.

Thus, each row in E amalgamates the entries in D corresponding with the states of the equivalence class i, therefore $ED = M_1$. □

Theorem 3 depends only on the in-amalgamations transforming M into M_1, thus the same procedure is defined for transforming the matrix M_i into M_{i+1}, obtaining the following sequence:

$$\begin{array}{cc} M = D_0E_0 & E_0D_0 = M_1 \\ M_1 = D_1E_1 & E_1D_1 = M_2 \\ & \vdots \\ M_{n-1} = D_{n-1}E_{n-1} & E_{n-1}D_{n-1} = M_n \end{array} \tag{3}$$

where M_n is the matrix representing the full shift of k symbols, hence theorem 3 has the following consequence:

Corollary 3. *Reversible automata $\mathcal{A} = (k, 2, \varphi)$ with Welch index $L = 1$ are strong shift equivalent to the full shift of k symbols with maximum lag $k - 1$.*

There are analogous results for reversible automata with Welch index $R = 1$ exchanging the position of D_i and E_i in equation (3) for $0 \leqslant i \leqslant n - 1$. An important point is that theorem 3 establishes a relevant restriction for checking whether the automaton $\mathcal{A}$ is reversible; the matrix M and the consequent matrices $M_1 \ldots M_{n-1}$ must have at least two identical columns and if a, b are two identical columns then each entry in row a is different from the same entry in row b.

6. Example

In this section we present an example with the automaton $\mathcal{A} = (4, 2, \varphi)$; the evolution rule φ is presented by the matrix M showed in table 1.

For this automaton, the ancestors of each sequence in K^3 show Welch indices $L = 1$ and $R = 4$. With them we can define the inverse rule φ^{-1}, take $w \in K^3$, $v \in K^4$ such that $\varphi(v) = w, a \in K$ and $u \in K^3$ such that $au = v$; then $\varphi^{-1}(w) = a$ where a is placed on the left end of w going backwards in the evolution of the automaton. Thus, the Welch index

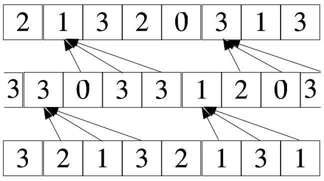

Figure 4. Inverse evolution of the automaton $\mathcal{A} = (4, 2, \varphi)$; the initial configuration is a ring of eight cells and for simplicity only some inverse evolutions are presented.

Table 2. Inverse rule of the reversible automaton presented in table 1.

w	φ^{-1} (**w**)	**w**	φ^{-1} (**w**)	**w**	φ^{-1} (**w**)	**w**	φ^{-1} (**w**)
000	0	100	1	200	3	300	2
001	0	101	1	201	3	301	2
002	0	102	1	202	3	302	2
003	0	103	1	203	3	303	2
010	0	110	1	210	3	310	2
011	0	111	1	211	3	311	2
012	0	112	1	212	3	312	2
013	0	113	1	213	3	313	2
020	1	120	0	220	2	320	3
021	1	121	0	221	2	321	3
022	0	122	1	222	2	322	3
023	0	123	1	223	2	323	3
030	0	130	1	230	2	330	3
031	0	131	1	231	2	331	3
032	1	132	0	232	2	332	3
033	1	133	0	233	2	333	3

$L = 1$ defines the inverse evolution rule at the left side for the sequences in K^3. Table 2 shows the inverse evolution of each $w \in K^3$.

An example of the inverse evolution of the automaton is presented in figure 4, where the evolutions of some inverse neighbourhoods are depicted.

The procedure for reviewing the existence of indirect paths in **D** yields the following results for the first row of M:

(i) Entry $(0, 1) = 0$ in M, therefore there is an indirect path from the self-loop 0 to the self-loop 1.
(ii) Entry $(1, 3) = 0$ in row 1, and because there is an indirect path from the self-loop 0 to the self-loop 1, there is also an indirect path from the self-loop 0 to the self-loop 3.
(iii) As $(0, 2) = 0$, an indirect path is defined from the self-loop 0 to the self-loop 2, it is the only indirect path defined in this step because row 2 has no element equal to 0.
(iv) As $(0, 3) = 1$, there is not another indirect path defined in this step.

Following the same process for the other rows, we obtain that **D** has a single indirect path connecting each ordered pair of self-loops. In order to know if $\mathcal{A}$ is reversible, let us apply the in-amalgamation procedure over the columns of M. Table 1 shows that columns 0 and 1 are equal, therefore they will be amalgamated for obtaining a new edge shift; repeating the procedure we get the sequence of in-amalgamations transforming M into the full shift of k symbols (table 3).

The graph representation of this transformation is depicted in figure 5.

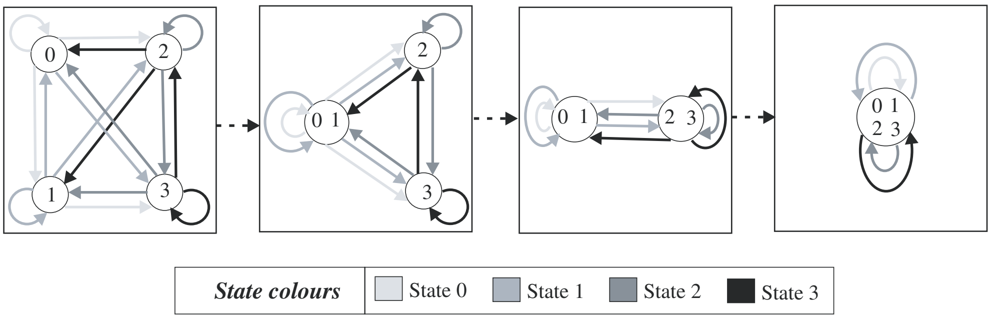

Figure 5. In-amalgamations transforming **D** into the full shift of four elements.

Table 3. Sequence of in-amalgamations transforming M into the full shift.

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3\\ \hline 0 & 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 1 & 0\\ 2 & 3 & 3 & 2 & 2\\ 3 & 2 & 2 & 3 & 3\end{array} \rightarrow \begin{array}{c|ccc} & 0,1 & 2 & 3\\ \hline 0,1 & 0,1 & 0,1 & 0,1\\ 2 & 3 & 2 & 2\\ 3 & 2 & 3 & 3\end{array} \rightarrow \begin{array}{c|cc} & 0,1 & 2,3\\ \hline 0,1 & 0,1 & 0,1\\ 2,3 & 3,2 & 2,3\end{array} \rightarrow \begin{array}{c|c} & 0,1,2,3\\ \hline 0,1,2,3 & 0,1,2,3\end{array}$$

Table 4. Sequence of matrix products transforming M into the full shift of four elements.

$$\begin{pmatrix}0&0&0&1\\1&1&1&0\\3&3&2&2\\2&2&3&3\end{pmatrix} = \begin{pmatrix}0&0&1\\1&1&0\\3&2&2\\2&3&3\end{pmatrix}\begin{pmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} \qquad \begin{pmatrix}1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\1&1&0\\3&2&2\\2&3&3\end{pmatrix} = \begin{pmatrix}0,1&0,1&1,0\\3&2&2\\2&3&3\end{pmatrix}$$

$$\begin{pmatrix}0,1&0,1&1,0\\3&2&2\\2&3&3\end{pmatrix} = \begin{pmatrix}0,1&0,1\\3&2\\2&3\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\end{pmatrix} \qquad \begin{pmatrix}1&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}0,1&0,1\\3&2\\2&3\end{pmatrix} = \begin{pmatrix}0,1&0,1\\3,2&3,2\end{pmatrix}$$

$$\begin{pmatrix}0,1&0,1\\3,2&3,2\end{pmatrix} = \begin{pmatrix}0,1\\3,2\end{pmatrix}\begin{pmatrix}1&1\end{pmatrix} \qquad \begin{pmatrix}1&1\end{pmatrix}\begin{pmatrix}0,1\\3,2\end{pmatrix} = (0,1,3,2)$$

By means of the procedure described in the proof of theorem 3, we obtain the sequence of matrix products transforming M into the full shift (table 4).

Therefore, $\mathcal{A}$ and the full shift of four elements are strong shift equivalent with lag 2.

7. Concluding remarks

The classification of the paths connecting the self-loops in a reversible automaton is useful to understand how the local behaviour which is not reversible is able to yield an invertible global behaviour; in this paper we have fully characterized these paths as indirect ones for reversible automata with Welch index $L = 1$.

For a reversible automaton, the matrix representation of the evolution rule is very useful for detecting if a given cellular automaton is reversible and strong shift equivalent with the full shift, the basis of this result is the relevant work developed by Kari defining equivalence relations in accordance with the left extensions of the ancestors in the automaton with a Welch index 1.

By means of in-amalgamations, it is possible to define a procedure for enumerating reversible automata with Welch index $L = 1$ for a given number of states, we just have to check the existence of identical columns for obtaining the matrix representation of each in-amalgamation.

A suggested work is to extend this analysis for a reversible automaton with any possible Welch indices; this study will be useful for knowing the kinds of paths connecting the self-loops in the corresponding de Bruijn diagram and for characterizing its relation with the full shift.

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